A new source of seed pairs for Golay sequences of length 2^m

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Abstract

In 2007 Jedwab and Parker proposed [10] that the natural viewpoint for a Golay complementary sequence is as a projection of a multi-dimensional Golay array. In 2008 Fiedler, Jedwab and Parker [5] used this viewpoint to show how to construct and enumerate all known 2^h-phase Golay sequences of length 2^m, starting from two sources of Golay seed pairs. The first source of seed pairs is the trivial Golay pair of length 1, which gives rise to "standard" Golay sequences; the second source is the set of 512 non-standard "cross-over" 4-phase Golay pairs of length 8, which give rise to non-standard 4-phase Golay sequences of length 2^m for each $m \geq 4$. Beginning with a single length 5 complex-valued Golay sequence pair, we show how to construct a third source of Golay seed pairs (and only the second known non-trivial source), namely a new set of 5184 non-standard 6-phase Golay sequences of length 16 that form 62208 non-standard ordered Golay pairs. Using the multi-dimensional viewpoint, this new set of Golay seed pairs in turn gives rise to a new infinite family of 6-phase non-standard Golay sequences of length 2^m for each $m \geq 4$, and a new infinite family of 12-phase non-standard Golay sequences of length 2^m for each $m \geq 8$. All currently known H-phase Golay sequences of length 2^m can be constructed from the three sets of seed pairs.

Keywords 6-phase, complementary, construction, Golay sequence, non-standard, seed pair

1 Introduction

Golay complementary sequence pairs have found application in many areas of digital information processing since their introduction by Golay [6] in 1951, including infrared multislit spectrometry [6], optical time domain reflectometry [12], power control for multicarrier wireless transmission [1], and medical ultrasound [13] (see [9], for example, for a general discussion). The central theoretical questions are: for what lengths does a Golay sequence pair exist, and how many distinct Golay sequences and Golay sequence pairs of a given length are there?

In 1999 Davis and Jedwab [1] gave an explicit algebraic normal form for a set of H-phase Golay sequence pairs of length 2^m in the case $H = 2^h$, demonstrating an unexpected connection with Reed-Muller codes; the same construction holds without modification for any even $H \neq 2^h$ [14]. In 2008 Fiedler, Jedwab and Parker [5] showed that all these "standard" Golay sequences can be recovered from a three-stage multi-dimensional construction process, using trivial Golay pairs of length 1 as inputs.

Until now, the only known non-standard H-phase Golay sequences of length 2^m were those arising when one or more "cross-over" 4-phase Golay pairs of length 8 [3] are used as inputs to

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the three-stage construction process [5]. In particular, this includes the 1024 non-standard 4-phase Golay sequences of length 16 found in 2005 by Li and Chu [11], that motivated [3] and subsequent work. It is then natural to ask whether there are any further sources of H-phase Golay seed pairs of length 2^m, that can be used as inputs to the three-stage construction process of [5] in order to construct new infinite families of Golay sequences.

The smallest length 2^m for which an exhaustive search of 4-phase Golay sequences has not been conducted is 32 [4, Table 1], but it appears that such a search remains out of computational reach using current search algorithms. However we discovered by exhaustive search that there are exactly 5184 non-standard 6-phase Golay sequences of length 16 that form 62208 non-standard ordered Golay pairs. Since there are no cross-over 6-phase Golay sequence pairs of length 8, we cannot explain the origin of these new sequences by similar methods to those of [3]; instead, an entirely different explanation is required. We give a preliminary classification of the 5184 sequences into two classes, each having the same "template". We then describe how the template led us to the desired complete explanation of the origin of the 5184 sequences, depending on the existence of only a single length 5 complex-valued Golay sequence pair. We outline the infinite families of Golay sequences that arise by using one or more of the newly discovered non-standard sequence pairs as inputs to the three-stage construction process. Finally, we compare our construction method for 6-phase Golay sequence pairs with a construction for binary Golay sequence pairs given in 1991 by Eliahou, Kervaire and Saffari [2].

2 Definitions and notation

We define a *length s sequence* to be a 1-dimensional matrix $\mathcal{A} = (A_i)$ of complex-valued entries, where i is integer, for which

$$A_i = 0 \quad \text{if } i < 0 \text{ or } i \geq s.$$

Call the set of sequence elements

$$\{A_i \mid 0 \leq i < s\}$$

the *in-range entries* of $\mathcal{A}$.

Usually the in-range entries of $\mathcal{A}$ are constrained to lie in a small finite set S called the sequence *alphabet*. Let ξ be a primitive H-th root of unity for some H, where H represents an even integer throughout. If $S = \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}$ then $\mathcal{A}$ is an *H-phase* sequence. Special cases of interest are the *binary* case $H = 2$, for which $S = \{1, -1\}$, and the *quaternary* case $H = 4$, for which $S = \{1, \sqrt{-1}, -1, -\sqrt{-1}\}$. If $S = \mathbb{Z}_H$ then $\mathcal{A}$ is a *sequence over $\mathbb{Z}_H$*. The in-range entries of an H-phase sequence $\mathcal{A} = (A_i)$ of length s can be represented in the form

$$\xi^{a_i} := A_i, \text{ where each } a_i \in \mathbb{Z}_H. \tag{1}$$

We say that the length s sequence (a_i) given by (1) is the *sequence over $\mathbb{Z}_H$ corresponding to $\mathcal{A}$*. (Here and elsewhere, in defining the elements of a sequence of a given length, the definition implicitly applies only to the in-range entries.) We will consistently use lower-case letters for sequences over $\mathbb{Z}_H$ ("additive notation"), and upper-case letters for complex-valued sequences ("multiplicative notation"). We will switch between multiplicative and additive notation, according to convenience. If an H-phase alphabet is enlarged to allow zero elements, so that $S = \{0, 1, \xi, \xi^2, \ldots, \xi^{H-1}\}$, then we call $\mathcal{A}$ an *$H^{(0)}$-phase* sequence. The *ternary* case $H = 2$, for which $S = \{0, 1, -1\}$, has been particularly studied.

The *aperiodic autocorrelation function* of a length s complex-valued sequence $\mathcal{A} = (A_i)$ is given by

$$C_{\mathcal{A}}(u) := \sum_i A_i \overline{A_{i+u}} \quad \text{for integer } u,$$

where bar represents complex conjugation. The aperiodic autocorrelation function of a sequence over $\mathbb{Z}_H$ is that of the corresponding H-phase sequence. A length s *Golay sequence pair* is a pair of length s sequences $\mathcal{A}$ and $\mathcal{B}$ for which

$$C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u) = 0 \quad \text{for all } u \neq 0.$$

A sequence $\mathcal{A}$ is called a *Golay sequence* if it forms a Golay sequence pair with some sequence $\mathcal{B}$.

The *algebraic normal form* of a sequence (a_i) of length 2^m over $\mathbb{Z}_H$ is the unique function $a'(i_1, \ldots, i_m) : \mathbb{Z}_2^m \to \mathbb{Z}_H$ satisfying

$$a'(i_1, \ldots, i_m) = a_{i_m + 2i_{m-1} + \cdots + 2^{m-1} i_1} \quad \text{for all } (i_1, \ldots, i_m) \in \mathbb{Z}_2^m.$$

We can explicitly represent a large class of Golay sequences of length 2^m using algebraic normal form:

Theorem 1. *Let $m \geq 1$ be integer and let $e'_0, e_0, e_1, \ldots, e_m \in \mathbb{Z}_H$. For any permutation π of $\{1, \ldots, m\}$, the sequences of length 2^m over $\mathbb{Z}_H$ having algebraic normal form*

$$\left.\begin{aligned} a(x_1, \ldots, x_m) &:= \frac{H}{2} \sum_{k=1}^{m-1} x_{\pi(k)} x_{\pi(k+1)} + \sum_{k=1}^{m} e_k x_{\pi(k)} + e_0, \\ b(x_1, \ldots, x_m) &:= \frac{H}{2} \sum_{k=1}^{m-1} x_{\pi(k)} x_{\pi(k+1)} + \sum_{k=1}^{m} e_k x_{\pi(k)} + e'_0 + \frac{H}{2} x_{\pi(1)} \end{aligned}\right\} \tag{2}$$

form a Golay sequence pair. As $e'_0, e_0, e_1, \ldots, e_m$ and π range over all their possible values, the number of Golay sequences of length 2^m over $\mathbb{Z}_H$ of this form is

$$\begin{cases} H^{m+1} m!/2 & \text{for } m > 1 \\ H^2 & \text{for } m = 1, \end{cases}$$

and the corresponding number of ordered Golay sequence pairs is at least $H^{m+2} m!$

The case $H = 2^h$ of Theorem 1 was given by Davis and Jedwab [1, Theorem 3], and Paterson [14] showed that the case $H \neq 2^h$ holds without modification to the construction in [1]. The Golay sequence pairs of Theorem 1 are called *standard*. The standard Golay sequences belong to a generalisation over $\mathbb{Z}_H$ of the binary Reed-Muller code [1].

Given a complex-valued length s sequence $\mathcal{A} = (A_i)$ and complex constant C, define $\mathcal{A}^*$ to be the length s sequence $(\overline{A_{s-1-i}})$ (with corresponding sequence $(-a_{s-1-i})$ over $\mathbb{Z}_H$ if $\mathcal{A}$ is H-phase), and $C\mathcal{A}$ to be the length s sequence (CA_i). The following result is a straightforward consequence of the definitions:

Lemma 2. *Let $\mathcal{A}$ be a complex-valued sequence of length s and let C be a complex constant of modulus 1. Then the sequences $\mathcal{A}$, $C\mathcal{A}$, and $\mathcal{A}^*$ have identical aperiodic autocorrelation function.*

Given an H-phase sequence $\mathcal{A}$ and a primitive H-th root of unity ξ, it follows from Lemma 2 that the elements of the set

$$E(\mathcal{A}) := \{\xi^c \mathcal{A} \mid c \in \mathbb{Z}_H\} \cup \{\xi^c \mathcal{A}^* \mid c \in \mathbb{Z}_H\}$$

of H-phase sequences (which has order H if $\mathcal{A}^* = \xi^c \mathcal{A}$ for some $c \in \mathbb{Z}_H$, and order $2H$ otherwise) all have identical aperiodic autocorrelation function.

If there are standard Golay sequence pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ of the same length for which $E(\mathcal{A}) \neq E(\mathcal{A}')$ and $E(\mathcal{B}) \neq E(\mathcal{B}')$ but $\mathcal{A}$, $\mathcal{A}'$ have identical aperiodic autocorrelation function, then we can form *cross-over* Golay sequence pairs $(\mathcal{A}, \mathcal{B}')$ and $(\mathcal{A}', \mathcal{B})$. The only known examples of cross-over Golay sequence pairs of length 2^m over $\mathbb{Z}_{2^h}$ occur for $m = 3$ and $h = 2$, for example the sequences $[0, 0, 0, 2, 0, 0, 2, 0]$ and $[0, 1, 1, 2, 0, 3, 3, 2]$ over $\mathbb{Z}_4$ [3]. All 512 ordered cross-over Golay sequence pairs of length 8 over $\mathbb{Z}_4$ can be derived from this pair, and all previously known non-standard Golay sequences of length 2^m over $\mathbb{Z}_H$ arise by using one or more of these cross-over sequence pairs as inputs to the three-stage construction process of [5].

3 How do the new 6-phase Golay sequences arise?

By exhaustive search, the total number of 6-phase Golay sequences of length 16 is 98496, whereas by Theorem 1 only $6^5 \cdot 4!/2 = 93312$ of these are standard; so there are 5184 non-standard 6-phase Golay sequences of length 16. Each of these non-standard Golay sequences is found to form a Golay pair with exactly 12 other non-standard Golay sequences, and with no standard Golay sequence. In summary, we find by computer that:

Proposition 3. *There are exactly* 5184 *non-standard* 6*-phase Golay sequences of length* 16*, and the corresponding number of non-standard ordered Golay sequence pairs is exactly* 62208*.*

For example (using additive notation), the sequences

$$[0, 0, 0, 0, 4, 2, 3, 4, 2, 0, 4, 2, 3, 0, 0, 3] \text{ and } [0, 3, 0, 3, 1, 2, 0, 1, 2, 0, 1, 2, 3, 3, 0, 0]$$

form a Golay pair of length 16 over $\mathbb{Z}_6$. The non-standard Golay sequences of Proposition 3 do not arise via cross-over, since direct checking shows there are no cross-over 6-phase Golay pairs of length 8. How, then, do they arise?

We begin with a preliminary classification of the 5184 sequences into two classes, each following the same "template". This classification requires Lemmas 4 and 5, both of which describe the transformation of a given Golay pair into one or more further Golay pairs. These transformation lemmas are slight modifications of known results, the only difference being that an H-phase alphabet can be extended to an $H^{(0)}$-phase alphabet. The first transformation concerns "offset sequences" (that were called *affine offsets* using additive notation in [5]):

Lemma 4 (Fiedler and Jedwab [3, Corollary 2])**.** *Let ξ be a primitive H-th root of unity, and suppose that $\mathcal{A} = (A_i)$ and $\mathcal{B} = (B_i)$ form an $H^{(0)}$-phase Golay sequence pair. Then the offset sequences $\mathcal{A}' = (\xi^{ei+e_0} A_i)$ and $\mathcal{B}' = (\xi^{ei+e_0'} B_i)$ also form an $H^{(0)}$-phase Golay sequence pair of the same length, for all $e, e_0, e_0' \in \mathbb{Z}_H$.*

Proof. Clearly $\mathcal{A}'$ and $\mathcal{B}'$ are $H^{(0)}$-phase sequences. We also have

$$\begin{aligned} C_{\mathcal{A}'}(u) + C_{\mathcal{B}'}(u) &= \sum_i \xi^{ei+e_0} A_i \overline{\xi^{e(i+u)+e_0} A_{i+u}} + \sum_i \xi^{ei+e_0'} B_i \overline{\xi^{e(i+u)+e_0'} B_{i+u}} \\ &= \xi^{-eu} \left(\sum_i A_i \overline{A_{i+u}} + \sum_i B_i \overline{B_{i+u}} \right) \\ &= \xi^{-eu} (C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u)), \end{aligned}$$

which implies the result. □

The second transformation concerns complex conjugation. Given a complex-valued sequence $\mathcal{A} = (A_i)$, write $\overline{\mathcal{A}}$ for the complex conjugated sequence $(\overline{A_i})$ (with corresponding sequence $(-a_i)$ over $\mathbb{Z}_H$ if $\mathcal{A}$ is H-phase).

Lemma 5 (Holzmann and Kharaghani [8]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ form an $H^{(0)}$-phase Golay sequence pair. Then the complex conjugated sequences $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ also form an $H^{(0)}$-phase Golay sequence pair of the same length.*

Proof. The result follows from the easily verified identity

$$C_{\overline{\mathcal{A}}}(u) + C_{\overline{\mathcal{B}}}(u) = \overline{C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u)}.$$

□

We begin by reducing the non-standard 6-phase Golay sequences of length 16 to a more manageable set, from which we can identify relationships among the sequences. By Lemma 4, we can reduce the 5184 sequences to a set of size $5184/36 = 144$, by choosing one representative (A_i) from each set of 36 offset sequences $\{(\xi^{ei+e_0}A_i) \mid e, e_0 \in \mathbb{Z}_6\}$. By Lemmas 2 and 5, we obtain a further reduction to a set of $144/4 = 36$ sequences, by choosing one representative $\mathcal{A}$ from each set $\{\mathcal{A}, \overline{\mathcal{A}}, \mathcal{A}^*, \overline{\mathcal{A}^*}\}$. By examination of the aperiodic autocorrelation function of the resulting 36 sequences, and by careful adjustment of the choice of representatives, we were able to arrange 36 representative sequences into two classes of nine Golay pairs, each class displaying clear internal structure. This structure is most easily described using additive notation to represent the Golay sequences, writing $\mathcal{A}+\mathcal{B}$ for the elementwise sum of the sequences $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{Z}_6$, and $c\mathcal{A}$ for the elementwise product of $\mathcal{A}$ with the constant $c \in \mathbb{Z}_6$. Using this notation, the template of Figure 1 contains nine Golay pairs arranged in a 3×3 grid. The second and third rows are formed by adding the sequence $\mathcal{X}$ to both sequences of the pairs of the preceding row, and the second and third columns are formed by adding the sequence $\mathcal{Y}$ to both sequences of the pairs of the preceding column. The set of 18 sequences contained in the template is

$$\{\mathcal{W} + c_1\mathcal{X} + c_2\mathcal{Y} \mid c_1 \in \mathbb{Z}_6,\, c_2 \in \{0, 1, 2\}\}, \tag{3}$$

each of which is a representative of $4 \cdot 36 = 144$ Golay sequences. The two possible values of $(\mathcal{W}, \mathcal{X}, \mathcal{Y})$ shown in Figure 1, generating sequence classes 1 and 2, then account for all $2 \cdot 18 \cdot 144 = 5184$ Golay sequences. (For both classes, taking $c_2 \in \{3, 4, 5\}$ in (3) does not generate any Golay sequences beyond those already described.)

$\{\mathcal{W},\ \mathcal{W}+3\mathcal{X}\}$	$\{\mathcal{W}+\mathcal{Y},\ \mathcal{W}+\mathcal{Y}+3\mathcal{X}\}$	$\{\mathcal{W}+2\mathcal{Y},\ \mathcal{W}+2\mathcal{Y}+3\mathcal{X}\}$
$\{\mathcal{W}+\mathcal{X},\ \mathcal{W}+4\mathcal{X}\}$	$\{\mathcal{W}+\mathcal{Y}+\mathcal{X},\ \mathcal{W}+\mathcal{Y}+4\mathcal{X}\}$	$\{\mathcal{W}+2\mathcal{Y}+\mathcal{X},\ \mathcal{W}+2\mathcal{Y}+4\mathcal{X}\}$
$\{\mathcal{W}+2\mathcal{X},\ \mathcal{W}+5\mathcal{X}\}$	$\{\mathcal{W}+\mathcal{Y}+2\mathcal{X},\ \mathcal{W}+\mathcal{Y}+5\mathcal{X}\}$	$\{\mathcal{W}+2\mathcal{Y}+2\mathcal{X},\ \mathcal{W}+2\mathcal{Y}+5\mathcal{X}\}$

Class 1:

$$\begin{aligned}\mathcal{W} &= [0,0,0,0,4,2,3,4,2,0,4,2,3,0,0,3]\\ \mathcal{X} &= [0,1,0,1,1,0,1,1,0,0,1,0,0,1,0,1]\\ \mathcal{Y} &= [0,2,0,2,3,1,3,4,2,3,5,3,4,0,4,0]\end{aligned}$$

Class 2:

$$\begin{aligned}\mathcal{W} &= [0,2,2,2,0,0,0,3,0,3,3,0,4,4,4,0]\\ \mathcal{X} &= [0,0,0,0,0,0,1,1,0,0,1,1,1,1,1,1]\\ \mathcal{Y} &= [0,1,2,3,4,0,2,3,3,4,0,2,3,4,5,0]\end{aligned}$$

Figure 1: Template for the 5184 non-standard Golay sequences of length 16 over $\mathbb{Z}_6$, and sequence values for Classes 1 and 2

The template indicates that there is striking structure among the 5184 sequences, but does not provide a complete explanation for their origin. For both classes 1 and 2, it is not obvious why $\{\mathcal{W}, \mathcal{W} + 3\mathcal{X}\}$ forms a Golay pair. Furthermore, although we can explain from [3, Lemma 3] why repeated addition of $\mathcal{X}$ to both sequences of the pair $\{\mathcal{W}, \mathcal{W} + 3\mathcal{X}\}$ yields another Golay pair, it is not clear why the same holds for addition of $\mathcal{Y}$. However the additive structure of the template strongly suggests that offset sequences play a key role in explaining the origin of the 5184 non-standard Golay sequences. The missing ingredient is the following construction for complex-valued Golay sequences over an arbitrary alphabet, for which we now revert to multiplicative notation. We write $\mathcal{A} \pm \mathcal{B}$ for the elementwise sum or difference of the complex-valued sequences $\mathcal{A}$ and $\mathcal{B}$.

Lemma 6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ form a complex-valued Golay sequence pair. Then $\mathcal{A}+\mathcal{B}$ and $\mathcal{A}-\mathcal{B}$ also form a Golay sequence pair of the same length.*

Proof. The result follows from the easily verified identity

$$C_{\mathcal{A}+\mathcal{B}}(u) + C_{\mathcal{A}-\mathcal{B}}(u) = 2(C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u)).$$

□

In general, the alphabet of the constructed sequences $\mathcal{A}+\mathcal{B}$ and $\mathcal{A}-\mathcal{B}$ in Lemma 6 will not be the same as that of $\mathcal{A}$ and $\mathcal{B}$. However, we can control the alphabet of the output sequences under two successive applications of Lemma 6, by careful choice of the input sequences.

Let $\xi = \exp(2\pi\sqrt{-1}/6)$ be a primitive sixth root of unity. It is easy to check that the sequences

$$\mathcal{F} = (F_i) := [1, \xi^2, \xi^2, \xi^2, 1] \text{ and } \mathcal{G} = (G_i) := [1, 0, 0, 1, -1]$$

form a $6^{(0)}$-phase Golay sequence pair of length 5. We now construct all 5184 non-standard 6-phase Golay sequences of length 16 from the pair $(\mathcal{F}, \mathcal{G})$. For any fixed e, e_0, e'_0 in $\mathbb{Z}_6$, by Lemma 4

$$\mathcal{A} = (A_i) := (\xi^{ei+e_0} F_i) \quad \text{and } \mathcal{B} = (B_i) := (\xi^{ei+e'_0} G_i) \tag{4}$$

form a $6^{(0)}$-phase Golay pair of length 5. Clearly, we can pad the sequences $\mathcal{A}$ and $\mathcal{B}$ with zeroes to form the $6^{(0)}$-phase Golay pair $(\mathcal{A}_1, \mathcal{B}_1)$ of length 16 shown in Figure 2 (noting that $B_1 = B_2 = 0$ by definition of $\mathcal{G}$). Now by Lemma 6,

$$\mathcal{A}_2 := \mathcal{A}_1 + \mathcal{B}_1 \text{ and } \mathcal{B}_2 := \mathcal{A}_1 - \mathcal{B}_1 \tag{5}$$

then form a Golay pair of length 16, and this pair is also $6^{(0)}$-phase because the support of $\mathcal{A}_1$ (namely the values of i for which element i of $\mathcal{A}_1$ is nonzero) is disjoint from that of $\mathcal{B}_1$. By Lemma 2, $\mathcal{A}_2$ therefore forms a $6^{(0)}$-phase Golay pair with $C\mathcal{B}_2^*$, where $C = \xi^c$ for any fixed $c \in \mathbb{Z}_6$. Moreover the indices $\{0, 1, \ldots, 15\}$ are the disjoint union of the support of $\mathcal{A}_2$ and $C\mathcal{B}_2^*$; in other words (as a consequence of the positions of the zero elements of $\mathcal{G}$), the sequences $\mathcal{A}_2$ and $C\mathcal{B}_2^*$ "fit together" perfectly, with the positions for which each is nonzero coinciding exactly with the positions for which the other is zero. So, using Lemma 6 again, we find that the length 16 Golay pair

$$\mathcal{A}_3 := \mathcal{A}_2 + C\mathcal{B}_2^* \text{ and } \mathcal{B}_3 := \mathcal{A}_2 - C\mathcal{B}_2^* \tag{6}$$

is 6-phase. The number of distinct non-standard 6-phase Golay sequences of length 16 having the form of $\mathcal{A}_3$ is 6^4, since each of e, e_0, e'_0, c can take any value in $\mathbb{Z}_6$. (We cannot increase this number by considering sequences of the form $\mathcal{B}_3$, since each $\mathcal{B}_3$ transforms to its pair $\mathcal{A}_3$ under the mapping $c \mapsto c+3$.)

We next examine how these 6^4 sequences form Golay pairs. From Section 2, we know that $\mathcal{A}_3$ forms a Golay pair with each of the sequences in the set

$$E(\mathcal{B}_3) := \{\xi^d \mathcal{B}_3 \mid d \in \mathbb{Z}_6\} \cup \{\xi^d \mathcal{B}_3^* \mid d \in \mathbb{Z}_6\}, \tag{7}$$

$$
\begin{array}{rcccccccccccccccc}
\mathcal{A}_1 = [& 0 & 0 & A_0 & 0 & 0 & A_1 & 0 & 0 & A_2 & 0 & 0 & A_3 & 0 & 0 & A_4 & 0 \;] \\
\mathcal{B}_1 = [& B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & B_4 & 0 & 0 & 0 \;] \\
\\
\mathcal{A}_2 = [& B_0 & 0 & A_0 & 0 & 0 & A_1 & 0 & 0 & A_2 & B_3 & 0 & A_3 & B_4 & 0 & A_4 & 0 \;] \\
\mathcal{B}_2 = [& -B_0 & 0 & A_0 & 0 & 0 & A_1 & 0 & 0 & A_2 & -B_3 & 0 & A_3 & -B_4 & 0 & A_4 & 0 \;] \\
\\
\mathcal{A}_3 = [& B_0 & C\overline{A_4} & A_0 & -C\overline{B_4} & C\overline{A_3} & A_1 & -C\overline{B_3} & C\overline{A_2} & A_2 & B_3 & C\overline{A_1} & A_3 & B_4 & C\overline{A_0} & A_4 & -C\overline{B_0} \;] \\
\mathcal{B}_3 = [& B_0 & -C\overline{A_4} & A_0 & C\overline{B_4} & -C\overline{A_3} & A_1 & C\overline{B_3} & -C\overline{A_2} & A_2 & B_3 & -C\overline{A_1} & A_3 & B_4 & -C\overline{A_0} & A_4 & C\overline{B_0} \;]
\end{array}
$$

Figure 2: Iterative construction of Class 1 Golay sequences

and this set has order $2 \cdot 6 = 12$ since (by inspection of $\mathcal{B}_3$) we have $\mathcal{B}_3^* \neq \xi^d \mathcal{B}_3$ for any $d \in \mathbb{Z}_6$. In fact, each of the 12 sequences in the set (7) occurs as one of the 6^4 sequences already constructed: for each $d \in \mathbb{Z}_6$, the sequence $\mathcal{B}_3$ transforms to $\xi^d \mathcal{B}_3$ under the mapping

$$e_0 \mapsto e_0 + d, \quad e_0' \mapsto e_0' + d, \quad c \mapsto 2d + c; \tag{8}$$

and transforms to $\xi^d \mathcal{B}_3^*$ under the mapping

$$e_0 \mapsto e_0 + 3 + d - c, \quad e_0' \mapsto e_0' + d - c, \quad c \mapsto 2d - c. \tag{9}$$

Furthermore, by Lemma 2 we can replace $\mathcal{F}$ throughout by $\mathcal{F}^* = [1, \xi^4, \xi^4, \xi^4, 1]$ to obtain another 6^4 non-standard Golay sequences, each of which forms a Golay pair with 12 others from this second set of 6^4 sequences. We have thereby accounted for a total of $2 \cdot 6^4 = 2592$ of the Golay sequences of Proposition 3, and shown that each forms a Golay pair with at least (and, by computer search, exactly) 12 other Golay sequences from Proposition 3.

(These 2592 sequences are exactly the Class 1 sequences of Figure 1, and we can determine the values of $\mathcal{X}$ and $\mathcal{Y}$ from the constructed forms for $\mathcal{A}_3$ and $\mathcal{B}_3$. In particular, the componentwise difference between the sequences of each Golay pair of the template is $3\mathcal{X}$; for the Class 1 sequences this corresponds to the 6-phase sequence $[1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, 1, 1, -1, 1, -1]$, which is the componentwise quotient of $\mathcal{B}_3$ and $\mathcal{A}_3$. The mapping $c \mapsto c + 1$ corresponds to the transformation that adds the sequence $\mathcal{X}$ to both sequences of a Golay pair of the template, and the mapping $e \mapsto e + 1$ likewise corresponds via (4) to addition of the sequence $\mathcal{Y}$.)

The Class 2 sequences arise from a modification of the construction of Figure 2, as shown in Figure 3. The sequences $\mathcal{A}$ and $\mathcal{B}$, as defined in (4), can be padded with zeroes to form the $6^{(0)}$-phase Golay pair $(\mathcal{A}_1, \mathcal{B}_1)$ of length 16 shown in Figure 3. The construction of the Golay pairs $(\mathcal{A}_2, \mathcal{B}_2)$ and $(\mathcal{A}_3, \mathcal{B}_3)$ then follows (5) and (6), as before. $\mathcal{F}$ can again be replaced by $\mathcal{F}^*$, and the same mappings (8) and (9) apply. This accounts for the remaining $2 \cdot 6^4 = 2592$ Golay sequences of Proposition 3 and their pairings. The value of $\mathcal{X}$ and $\mathcal{Y}$ can be determined from the same mapping of c and e as previously.

$$
\begin{array}{rcccccccccccccccc}
\mathcal{A}_1 = [& A_0 & A_1 & A_2 & A_3 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \;] \\
\mathcal{B}_1 = [& 0 & 0 & 0 & 0 & 0 & B_0 & 0 & 0 & B_3 & B_4 & 0 & 0 & 0 & 0 & 0 & 0 \;] \\
\\
\mathcal{A}_2 = [& A_0 & A_1 & A_2 & A_3 & A_4 & B_0 & 0 & 0 & B_3 & B_4 & 0 & 0 & 0 & 0 & 0 & 0 \;] \\
\mathcal{B}_2 = [& A_0 & A_1 & A_2 & A_3 & A_4 & -B_0 & 0 & 0 & -B_3 & -B_4 & 0 & 0 & 0 & 0 & 0 & 0 \;] \\
\\
\mathcal{A}_3 = [& A_0 & A_1 & A_2 & A_3 & A_4 & B_0 & -C\overline{B_4} & -C\overline{B_3} & B_3 & B_4 & -C\overline{B_0} & C\overline{A_4} & C\overline{A_3} & C\overline{A_2} & C\overline{A_1} & C\overline{A_0} \;] \\
\mathcal{B}_3 = [& A_0 & A_1 & A_2 & A_3 & A_4 & B_0 & C\overline{B_4} & C\overline{B_3} & B_3 & B_4 & C\overline{B_0} & -C\overline{A_4} & -C\overline{A_3} & -C\overline{A_2} & -C\overline{A_1} & -C\overline{A_0} \;]
\end{array}
$$

Figure 3: Iterative construction of Class 2 Golay sequences

4 Two new infinite families of non-standard Golay sequences

The three-stage construction process of [5] explicitly constructs a multi-dimensional H-phase Golay array pair from multiple input H-phase Golay sequence (or array) pairs, and then produces numerous H-phase Golay sequence pairs by taking affine offsets and "projecting" to lower dimensions. By taking one or more of the input Golay array pairs to be from Proposition 3 and all other input pairs to be trivial, we obtain a new infinite family of non-standard 6-phase Golay sequences of length 2^m for each $m \geq 4$.

We can also regard both the 512 ordered cross-over 4-phase Golay pairs of length 8 and the 62208 ordered Golay pairs of Proposition 3 as 12-phase sequences, and use at least one pair of each type as inputs to the three-stage construction process. This produces a further new infinite family of non-standard 12-phase Golay sequences of length 2^m for each $m \geq 8$.

5 Comparison with an iterative binary construction

Beginning with a suitable $H^{(0)}$-phase Golay pair $(\mathcal{A}_1, \mathcal{B}_1)$ of length s, the construction method illustrated in Figures 2 and 3 produces H-phase Golay sequence pairs $\mathcal{A}_3 := (\alpha_i)$ and $\mathcal{B}_3 := (\beta_i)$ (with $H = 6$ and $s = 16$), whose elements satisfy

$$\beta_i = D_i \alpha_i \text{ for all } i, \text{ where each } D_i \in \{1, -1\} \text{ and } D_i = -D_{s-1-i} \tag{10}$$

and

$$\alpha_{s-1-i} = D'_i C \overline{\alpha_i} \text{ for all } i, \text{ where each } D'_i \in \{1, -1\} \text{ and } C \in \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}. \tag{11}$$

Given any H-phase Golay sequence pair $\mathcal{A}_3 := (\alpha_i)$ and $\mathcal{B}_3 := (\beta_i)$ of length s satisfying (10) and (11), it is straightforward to show that the construction can be reversed: the sequence pair $(\mathcal{A}_1, \mathcal{B}_1)$ that is then defined by (5) and (6), which forms a Golay pair by Lemmas 2 and 6, is $H^{(0)}$-phase.

Now for a standard H-phase Golay sequence pair $(\mathcal{A}_3, \mathcal{B}_3)$ of length 2^m, it can be shown from (2) that conditions (10) and (11) hold for $(\alpha_i) = \mathcal{A}_3$ and $(\beta_i) = C'\mathcal{B}_3$ for some $C' \in \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}$. The same is not necessarily true for a non-standard H-phase Golay sequence pair $(\mathcal{A}_3, \mathcal{B}_3)$ of length 2^m: for example, consider the cross-over Golay sequence pair $\mathcal{A}_3 = [0, 0, 0, 2, 0, 0, 2, 0]$ and $\mathcal{B}_3 = [0, 1, 1, 2, 0, 3, 3, 2]$ over $\mathbb{Z}_4$. However, in the case of a binary Golay sequence pair $\mathcal{A}_3 := (\alpha_i)$ and $\mathcal{B}_3 := (\beta_i)$ of length s, Golay [7] established that

$$\alpha_i \beta_i = -\alpha_{s-1-i} \beta_{s-1-i} \text{ for all } i,$$

which implies that conditions (10) and (11) always hold in the binary case (with $C = 1$), whether or not the pair is standard and whether or not it has length 2^m. Therefore we can explain the existence of any binary Golay sequence pair $(\mathcal{A}_3, \mathcal{B}_3)$ by means of the ternary Golay sequence pair $(\mathcal{A}_1, \mathcal{B}_1)$ defined from (5) and (6) with $C = 1$, as noted previously by Eliahou, Kervaire and Saffari [2]. For example, following the derivation of [2, p. 249], we now explain the existence of all 32 binary Golay sequences of length 10 as depending on the single length 3 ternary Golay sequence pair

$$\mathcal{F} = (F_i) := [1, 1, -1] \text{ and } \mathcal{G} = (G_i) := [1, 0, 1].$$

By Lemma 4, the sequences

$$\mathcal{A} = (A_i) := ((-1)^{ei+e_0} F_i) \text{ and } \mathcal{B} = (B_i) := ((-1)^{ei+e'_0} G_i) \tag{12}$$

form a ternary Golay sequence pair for any fixed e, e_0, e_0' in $\mathbb{Z}_2$. We pad the sequences $\mathcal{A}$ and $\mathcal{B}$ with zeroes as shown in Figures 4 and 5 to form a ternary Golay pair $(\mathcal{A}_1, \mathcal{B}_1)$ of length 10, and then set

$$\mathcal{A}_2 := \mathcal{A}_1 + \mathcal{B}_1 \text{ and } \mathcal{B}_2 := \mathcal{A}_1 - \mathcal{B}_1,$$
$$\mathcal{A}_3 := \mathcal{A}_2 + \mathcal{B}_2^* \text{ and } \mathcal{B}_3 := \mathcal{A}_2 - \mathcal{B}_2^*.$$

This accounts for all 32 binary Golay sequences of length 10, by taking the sequences $\mathcal{A}_3$ and $\mathcal{B}_3$ from Figures 4 and 5 for all 2^3 values of e, e_0, e_0'.

$$\begin{array}{l}
\mathcal{A}_1 = [\begin{array}{rrrrrrrrrr} 0 & A_0 & 0 & 0 & A_1 & 0 & 0 & A_2 & 0 & 0 \end{array}] \\
\mathcal{B}_1 = [\begin{array}{rrrrrrrrrr} B_0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 \end{array}] \\
\\
\mathcal{A}_2 = [\begin{array}{rrrrrrrrrr} B_0 & A_0 & 0 & 0 & A_1 & 0 & B_2 & A_2 & 0 & 0 \end{array}] \\
\mathcal{B}_2 = [\begin{array}{rrrrrrrrrr} -B_0 & A_0 & 0 & 0 & A_1 & 0 & -B_2 & A_2 & 0 & 0 \end{array}] \\
\\
\mathcal{A}_3 = [\begin{array}{rrrrrrrrrr} B_0 & A_0 & A_2 & -B_2 & A_1 & A_1 & B_2 & A_2 & A_0 & -B_0 \end{array}] \\
\mathcal{B}_3 = [\begin{array}{rrrrrrrrrr} B_0 & A_0 & -A_2 & B_2 & A_1 & -A_1 & B_2 & A_2 & -A_0 & B_0 \end{array}]
\end{array}$$

Figure 4: First iterative construction of length 10 binary Golay sequences

$$\begin{array}{l}
\mathcal{A}_1 = [\begin{array}{rrrrrrrrrr} A_0 & A_1 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}] \\
\mathcal{B}_1 = [\begin{array}{rrrrrrrrrr} 0 & 0 & 0 & B_0 & 0 & B_2 & 0 & 0 & 0 & 0 \end{array}] \\
\\
\mathcal{A}_2 = [\begin{array}{rrrrrrrrrr} A_0 & A_1 & A_2 & B_0 & 0 & B_2 & 0 & 0 & 0 & 0 \end{array}] \\
\mathcal{B}_2 = [\begin{array}{rrrrrrrrrr} A_0 & A_1 & A_2 & -B_0 & 0 & -B_2 & 0 & 0 & 0 & 0 \end{array}] \\
\\
\mathcal{A}_3 = [\begin{array}{rrrrrrrrrr} A_0 & A_1 & A_2 & B_0 & -B_2 & B_2 & -B_0 & A_2 & A_1 & A_0 \end{array}] \\
\mathcal{B}_3 = [\begin{array}{rrrrrrrrrr} A_0 & A_1 & A_2 & B_0 & B_2 & B_2 & B_0 & -A_2 & -A_1 & -A_0 \end{array}]
\end{array}$$

Figure 5: Second iterative construction of length 10 binary Golay sequences

6 Conclusions

By computer search, we have discovered a set of 5184 non-standard 6-phase Golay sequence pairs of length 16 that do not arise by cross-over. We have shown how to construct these sequences from a single complex-valued Golay sequence pair of length 5. By using one or more of the new Golay sequence pairs in the construction process of [5], we obtain two new infinite families of non-standard Golay sequences of length 2^m. All currently known H-phase Golay sequences of length 2^m can be constructed from three sets of seed pairs: the new set given in Proposition 3, the cross-over 4-phase pairs of length 8, and a trivial pair of length 1.

The construction method for H-phase Golay sequence pairs that we have described is new for $H > 2$. It relies both on the position of the zero elements of the initial pair $(\mathcal{F}, \mathcal{G})$, and on the values of the non-zero elements. Can further examples of new H-phase Golay sequences of length 2^m be produced by this method?

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